Chapter 4: Lagrangian Formalism

Lagrange’s general formulation of mechanics in terms of variational principles shows that conservation of energy arises as a direct consequence of temporal symmetry - the invariance of physical laws under a time translation. Similarly, conservation of momentum arises from spatial symmetry – the invariance of physical laws under a spatial translation (ie, the freedom to choose the origin of our coordinate system arbitrarily)[1]. While Bohm’s model has the correct temporal and spatial symmetry, it does not contain the corresponding conservation laws. (This possibility is permitted by the loophole that the model is not derivable from a Lagrangian.) Consequently, since the model does not possess the usual linkage between symmetry and conservation, this feature should be examined critically as a possible deficiency.[2].

4.1 Lagrangian Formalism for Particle Motion

The Lagrangian formalism provides a general formulation of the laws governing the behaviour of mechanical systems[3]. In the case of a single particle, the action S is a functional of the entire trajectory, which may be described by the parameterisation [$\mathbf{x_0}$(t), $\mathbf{v}$(t)], where $\mathbf{v} \equiv d\mathbf{x_0}/dt$. The subscript "0" is inserted here to distinguish the point $\mathbf{x_0}$ at

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[1] Tsung-Dao L., *Particle Physics and Introduction to Field Theory*, in The World of Physics. Weaver J.H. Published by Simon and Schuster, New York (1987).

[2] Annandan and Brown view the situation as follows: "It is well known that the dynamics of particles and fields, in classical and quantum physics, may be described by the action principle. The space-time translational invariance of the action of the system under consideration implies that the energy and momentum of the system are conserved. This means that the different components of the system satisfy the action-reaction (AR) principle. But if the action does not have translational invariance, then we would say, rather than give up energy-momentum conservation, that there is some external influence on the system, so that the internal components of the system do satisfy the AR principle. This suggests that the AR principle is more fundamental than any other law of physics, as if it is a condition for the reality and being of entities in a physical theory." Annandan J. & Brown H.R., *On the Reality of Space-Time Geometry and the Wavefunction,* Foundations of Physics, Vol. 25, pp. 349-360 (1995).

[3] See, e.g., Landau L.D. & Lifshitz E.M., *Course of Theoretical Physics. Vol. 1: Mechanics,* 2nd Edition. Pergamon Press, Oxford, London. (1969).

which the particle is located at time t from arbitrary spatial points **x** at which field values $\phi(\mathbf{x})$ are to be specified. The quantity $\mathbf{v}(t)$ represents the particle's velocity. The action S is defined to be the time integral of the difference between the particle's kinetic and potential energies over the trajectory between the end points of the motion under consideration[4]

$$\text{Action} = S = \int_{t_1}^{t_2} \{KE[\mathbf{v}(t)] - PE[\mathbf{x}_0(t), \mathbf{v}(t)]\}\, dt \qquad [4\text{-}1]$$

The "principle of least action" provides a global description for the time evolution of the system by asserting that an object's trajectory between specified points over a certain time interval is that for which the action is minimised[5]. The integrand of the action function is called the Lagrangian $L[\mathbf{x_0}(t), \mathbf{v}(t)]$:

$$L = KE - PE \qquad [4\text{-}2]$$

It is a function of the particle's position and velocity.

One of the principal advantages of the Lagrangian formulation of mechanics arises from the fact that the Lagrangian is required to satisfy a number of symmetries, such as, for example translational and rotational invariance and Lorentz and Gauge invariance. While the Lagrangian describing a physical system is not unique, these restrictions and the physical properties of the system under consideration frequently serve to identify an appropriate Lagrangian from which the differential form of the system's equations of motion may be determined. In other words, the Lagrangian for a system can often be

[4] See, e.g., Feynman R.P, Leighton R.B. and Sands M., Ch. 19, *The Principle of Least Action*, in *The Feynman Lectures on Physics*, Vol. 2. Addison-Wesley, Reading Massachusetts (1994). This reference provides a very readable introduction to the material under consideration.
[5] Strictly speaking, the Lagrangian formulation develops under the assumption that the action takes on a stationary value for the correct trajectory.

guessed by imposing certain physical restrictions, which include the homogeneity of space and time.

The equations of motion may always be obtained from the Lagrangian by determining the conditions under which the action takes on a stationary value. This leads to the general result, for determining equations of particle motion, known as Lagrange's equations (i = 1, 2, 3):

$$\frac{d}{dt}\frac{\partial L}{\partial v^i} = \frac{\partial L}{\partial x_0^i} \qquad [4\text{-}3]$$

where v^i is defined by:

$$v^i \equiv \frac{dx_0^i}{dt} \qquad [4\text{-}4]$$

These equations determine the particle's trajectory.

For a single particle in a real scalar field $\phi(x)$, the equation of motion describing the particle's trajectory can be derived from the following Lagrangian:

$$L_{\text{particle}} = \frac{mv^2}{2} - q\phi(\mathbf{x}_0) \qquad [4\text{-}5]$$

4.2 Lagrangian Formalism for Fields

The Lagrangian formulation also provides a general description of the time evolution of fields. For a field defined by the function Ð($\mathbf{x}$,t), the development relies upon introducing a Lagrangian density å, which is a function of the field and its derivatives. The action function then has the form of an integral over all space and time of this Lagrangian density:

$$S_{\text{field}} = \int_{t_1}^{t_2} L_{\text{field}}\, dt = \int_{t_1}^{t_2}\int_{-\infty}^{\infty} \text{å}\; d^3x\, dt \qquad [4\text{-}6]$$

The principle of least action asserts that the field develops in such a way that its action integral over a specified time be minimised[6]. By performing variations in the field $\delta\phi(\mathbf{x},t)$ while imposing this requirement, a differential equation describing the field's time evolution may be deduced. When ϕ is a real scalar valued function, the general solution for the differential field equation is ($\mu = 0,1,2,3$):

$$\frac{\partial}{\partial x^{\mu}} \frac{\partial \breve{a}}{\partial(\partial_{\mu}\phi)} - \frac{\partial \breve{a}}{\partial \phi} = 0 \qquad [4\text{-}7]$$

This equation has been expressed in terms of the usual four-dimensional notation of relativity:

$$(x^0, x^1, x^2, x^3) \equiv (ct, x, y, z) \qquad [4\text{-}8a]$$

$$(x_0, x_1, x_2, x_3) \equiv (ct, -x, -y, -z) \qquad [4\text{-}8b]$$

$$[\partial_0, \partial_1, \partial_2, \partial_3] \equiv [\partial/\partial(ct), \partial/\partial x, \partial/\partial y, \partial/\partial z] \qquad [4\text{-}8c]$$

$$[\partial^0, \partial^1, \partial^2, \partial^3] \equiv [\partial/\partial(ct), -\partial/\partial x, -\partial/\partial y, -\partial/\partial z] \qquad [4\text{-}8d]$$

$$\partial_{\nu} \equiv \partial/\partial x^{\nu} \qquad [4\text{-}8e]$$

$$\partial^{\nu} \equiv \partial/\partial x_{\nu} \qquad [4\text{-}8f]$$

Also as usual, repeated indices denote summation, in accordance with the Einstein summation convention. For example, the repeated μ index on the derivatives in [4-7] expands as:

$$\frac{\partial}{\partial x^{\mu}} \frac{\partial}{\partial(\partial_{\mu}\phi)} = \frac{\partial}{\partial x^{0}} \frac{\partial}{\partial(\partial_{0}\phi)} + \frac{\partial}{\partial x^{1}} \frac{\partial}{\partial(\partial_{1}\phi)} + \frac{\partial}{\partial x^{2}} \frac{\partial}{\partial(\partial_{2}\phi)} + \frac{\partial}{\partial x^{3}} \frac{\partial}{\partial(\partial_{3}\phi)} \qquad [4\text{-}9]$$

[6] See Footnote 5 concerning minimisation and stationarity of the action function.

A free, massless scalar field may be described by the following relativistically invariant Lagrangian density[7]:

$$\text{å}_{\text{massless scalar field}} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) \qquad [4\text{-}10]$$

Applying equation [4-7] to this Lagrangian density yields the appropriate field equation for such a field:

$$\partial_\mu\partial^\mu\phi = 0 \qquad [4\text{-}11]$$

For a vector field, such as the electromagnetic 4-potential A^μ, equation [4-7] must be generalised to the following set of equations:

$$\frac{\partial}{\partial x^\mu}\frac{\partial\, \text{ă}}{\partial\,(\partial_\mu A^\nu)} - \frac{\partial\, \text{ă}}{\partial A^\nu} = 0 \qquad [4\text{-}12]$$

For example, the Lagrangian density for the free-field electromagnetic case is[8]:

$$\text{å}_{\text{em field}} = -\frac{F_{\mu\nu}\,F^{\mu\nu}}{16\,\pi} \qquad [4\text{-}13a]$$

where

$$F^{\mu\nu} \equiv \partial^\nu A^\mu - \partial^\mu A^\nu \qquad [4\text{-}13b]$$

and applying equation [4-12] to the Lagrangian Density [4-13] yields the field equation:

$$\partial_\mu F^{\mu\nu} = 0 \qquad [4\text{-}14]$$

For fields defined by complex variables, the field ϕ and its conjugate ϕ^* are treated as independent variables. The field equation for ϕ is obtained by varying ϕ^* in the Lagrangian density. Similarly, the field equation for ϕ^* is obtained by varying ϕ.

[7] See, e.g., p. 97 in Schweber S., Bethe H. & Hoffmann F., *Mesons and Fields*, Vol. 1. Row, Peterson & Co. (1955).
[8] See, e.g., p. 583 in Goldstein H., *Classical Mechanics,* 2nd Ed. Addison-Wesley, Massachusetts (1980).

4.3 Noether's Theorem & Conservation

Another advantage of using the Lagrangian formulation is that it provides a direct method for determining the energy-momentum tensor for a field. Consequently, it permits a complete specification of both the distribution and time rate of change of energy and momentum density throughout the field. Furthermore, Noether's theorem[9] states that overall energy and momentum conservation will hold in any physical system that can be described by a Lagrangian having no explicit dependence on the space or time coordinates.

4.4 Overall Lagrangian for a Particle & Field in Interaction

Our aim is to construct a Lagrangian formulation of Bohm's model, with a single Lagrangian density describing both the Bohmian particle and the field with which it interacts. The latter is a scalar field defined in terms of the wave-function $\psi(x)$. As a guide to obtaining such a Lagrangian density expression, we will examine the known cases of a classical particle interacting with either a scalar or vector field. In the scalar case, the following two Lagrangians (already introduced above) must be combined together in a consistent manner:

$$L_{\text{particle}} = \frac{mv^2}{2} - q\phi(\mathbf{x}_0) \qquad [4\text{-}5]$$

$$L_{\text{massless scalar field}} = \int_{-\infty}^{\infty} \text{å}_{\text{massless scalar field}} \; d^3x \qquad [4\text{-}15a]$$

$$= \int_{-\infty}^{\infty} \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) \; d^3x \qquad [4\text{-}15b]$$

[9] See, e.g., Sec. 12-7 in Goldstein H., *Classical Mechanics,* 2nd Ed. Addison-Wesley, Massachusetts (1980).

For convenience in the present development, only the field is being treated relativistically. The particle motion is described by the non-relativistic equation arising from the Lagrangian [4-5]. Using a delta function, equation [4-5] can be written equivalently as:

$$L_{particle} = \int_{-\infty}^{\infty} [\, \frac{mv^2}{2} - q\phi(\mathbf{x}) \,]\ \delta(\mathbf{x} - \mathbf{x}_0(t))\ d^3x \qquad [4\text{-}16a]$$

$$\equiv \int_{-\infty}^{\infty} [\, \frac{mv^2}{2} - q\phi(\mathbf{x}) \,]\ \rho\ d^3x \qquad [4\text{-}16b]$$

where

$$\rho \equiv \delta(\mathbf{x} - \mathbf{x}_0(t)) \qquad [4\text{-}17]$$

can be thought of as the distribution of the particle through space.

Equation [4-16b] provides the link for combining [4-5] and [4-15b] in a unified way to obtain an overall particle-field Lagrangian density. The combined Lagrangian is:

$$L_{system} = \int_{-\infty}^{\infty} \text{å}_{system}\ d^3x \qquad [4\text{-}18a]$$

$$= \int_{-\infty}^{\infty} \frac{1}{2}\, (\partial_\mu\phi)\ (\partial^\mu\phi) + [\, \frac{mv^2}{2} - q\phi(\mathbf{x}) \,]\ \rho\ d^3x \qquad [4\text{-}18b]$$

and so the corresponding Lagrangian density å_{system} is then:

$$\text{å}_{system} = \frac{1}{2}\, (\partial_\mu\phi)\ (\partial^\mu\phi) + [\, \frac{mv^2}{2} - q\phi(\mathbf{x}) \,]\ \rho \qquad [4\text{-}19]$$

The field equation deriving from the particle-field system Lagrangian density [4-19] by applying Lagrange's scalar field differential operator equation [4-7] is similar to that of the free scalar field [4-11] but with the source term ($- q\ \rho$) added:

$$\partial_\mu\partial^\mu\phi = - q\ \rho \qquad [4\text{-}20]$$

Similarly the Lagrangian density for an electromagnetic field in interaction with a particle is[10]:

$$\mathring{a} = -\frac{F_{\mu\nu}\,F^{\mu\nu}}{16\,\pi} + \tfrac{1}{2}mv^2\,\rho - [\,q\phi - \frac{q}{c}\,\mathbf{v}.\mathbf{A}\,]\,\rho \qquad [4\text{-}21]$$

The particle field system Lagrangian densities [4-19] and [4-21] share a number of common features:

- they both consist of the sum of a free-field term, a particle term and an interaction term,
- the particle is described by the same expression $\frac{1}{2}mv^2\rho$ in both equations,
- both lead to field equations containing a source term,
- both lead to energy and momentum conservation for the particle-field system.

4.5 Squires attempted Lagrangian Formulation of Bohmian Mechanics

Squires[11] has taken preliminary steps in an attempt to remove the non-conservation defect in Bohm's dynamics. His formulation focuses on describing the particle's motion via the first-order equation $d\mathbf{x}/dt = \mathbf{\nabla}S/m$ only, rather than in terms of the quantum potential Q. [With this potential left out, energy and momentum non-conservation are still evident in the form of the time dependence of E and p in the equations $E = -\,\partial S(\mathbf{x},t)/\partial t$ and $\mathbf{p} = \mathbf{\nabla}S(\mathbf{x},t)$.] Squires observes that a natural way of introducing the desired action-reaction into Bohm's model is to derive the model from a Lagrangian and he proceeds by proposing the following action function:

$$A = A_S + A_\lambda + kA_N \qquad [4\text{-}22]$$

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[10] See, e.g., p. 586 in Goldstein H., *Classical Mechanics,* 2nd Ed. Addison-Wesley, Massachusetts (1980).

where λ is a Lagrange multiplier, k is an arbitrary parameter and the separate terms making up this overall action are given explicitly by:

$$A_s = \int dt\, dx\, [\, \frac{i\hbar}{2} (\psi^* \partial_t \psi - \psi \partial_t \psi^*) - \frac{\hbar^2}{2m} (\nabla\psi^*)(\nabla\psi) - V\, \psi^* \psi \,] \qquad [4\text{-}23]$$

$$A_\lambda = \int dt\, \boldsymbol{\lambda} \,.\, [\, \mathbf{v} + \frac{i\hbar}{2m} \int dx\, (\, \frac{\nabla\psi}{\psi} - \frac{\nabla\psi^*}{\psi^*}\,)\, \delta(\mathbf{x} - \mathbf{x}_0) \,] \qquad [4\text{-}24]$$

$$A_N = \int dt\, [\, \frac{mv^2}{2} - V(\mathbf{x}_0) \,] \qquad [4\text{-}25]$$

Variation with respect to $\boldsymbol{\lambda}$, ψ, and $\mathbf{x_0}$, respectively, yields the equations:

$$\mathbf{v} = -\frac{1}{m} \mathrm{Re}\, (\frac{i\hbar\, \nabla\psi}{\psi})_{\mathbf{x}=\mathbf{x_0}} \qquad [4\text{-}26]$$

$$i\hbar\, \partial_t \psi = H\psi - \frac{i\hbar}{2m\, \psi^*}\, \boldsymbol{\lambda} \,.\, \nabla\delta(\mathbf{x} - \mathbf{x}_0) \qquad [4\text{-}27]$$

$$\partial_t \boldsymbol{\lambda} = \frac{i\hbar}{2m} \nabla\, [\boldsymbol{\lambda} \,.\, (\frac{\nabla\psi}{\psi} - \frac{\nabla\psi^*}{\psi^*})]_{\mathbf{x}=\mathbf{x_0}} - k(m\dot{\mathbf{v}} + \nabla V) \qquad [4\text{-}28]$$

Equation [4-27] is the Schrodinger equation with the addition of an extra term. This can be viewed as a source term and suggests that the particle might somehow be regarded as the source of the wave function, which produces the quantum force. The existence of the parameter k allows us to assign the magnitude of the source term arbitrarily.

Squires stated that "work is in progress on these equations". No further developments in his approach are known at this time. There are two points of concern with his formulation. One is that the condition $\mathbf{v} = \nabla S/m$, which arises from the unmodified Schrodinger equation and which remains part of Squires' model (see [4-26]), is actually not consistent with the modified Schrodinger equation [4-27] he introduces, assuming

[11] Squires E.J., *Some Comments on the de Broglie-Bohm Picture by an Admiring Spectator,* pp. 125-38 in *Waves and Particles in Light and Matter,* Edited by van der Merwe A. and Garuccio A. Plenum Press, New York and London (1994).

that a conserved probability density $\psi^*\psi$ is required. The other is that it is not clear how to obtain an energy-momentum tensor expression corresponding to Squires' Lagrangian, as would be needed to formulate energy and momentum conservation explicitly for the system.

As stated above, Squires' approach relates only to the minimalist version of Bohm's model characterised by the equation $d\mathbf{x}/dt = \nabla S/m$, without the introduction of the quantum potential Q. The view to be pursued in our subsequent development here is that the equations involving this potential are the more appropriate ones to use in attempting to reinstate conservation via a Lagrangian approach.